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TANGENT ALGEBRAIC SUBVARIETIES OF VECTOR FIELDS

JUAN B. SANCHO DE SALAS

ABSTRACT. An algebraic commutative group G is associated to any vector field D on a complete algebraic variety X. The group G acts on X and its orbits are the minimal subvarieties of X which are tangent to D. This group is computed in the case of a vector field on $\mathbb{P}_n$.

INTRODUCTION

Let X be an algebraic variety over an algebraically closed field k of characteristic 0 and let D be a vector field on X. A closed subvariety Y of X, defined by a sheaf of ideals I of $\mathcal{O}_X$, is said to be a tangent subvariety of D if $D(I) \subseteq I$. This condition implies that D induces a derivation of the sheaf $\mathcal{O}_Y = \mathcal{O}_X/I$, so that it defines a vector field on Y. The aim of this paper is to study the structure of the family of tangent subvarieties of a vector field D. At first sight it seems reasonable to conjecture that minimal tangent subvarieties of D form an algebraic family. Precisely:

Question: Does there exist a dense open set U in X and a surjective morphism of k-schemes $\pi : U \to Z$ whose fibres are just the minimal tangent subvarieties of D in U?

Unfortunately, the answer is negative in general (we shall give a counterexample in Section 5). However, it is affirmative in an interesting case, that of global vector fields on a complete algebraic variety. For these fields we prove the following result:

Theorem. *Let D be a vector field on an integral projective variety X. There exists a connected commutative algebraic subgroup $G \subseteq \mathbf{Aut}\, X$ and a G–invariant dense open set U in X such that:*

a) *Minimal tangent subvarieties of D on U are just orbits of G in U.*

b) *The quotient variety $\pi : U \to Z = U/G$ exists. Hence, the fibres of π are the minimal tangent subvarieties of D in U.*

c) *The sheaf $\mathcal{O}_Z$ coincides with the sheaf of first integrals of D; that is to say, for any open set V in Z, we have*

$$\Gamma(V, \mathcal{O}_Z) \;=\; \{f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0\}.$$

This theorem may be extended to a distribution on X generated by global vector fields (see 3.6).

The completeness of X is used to assure the existence of the algebraic group $\mathbf{Aut}\, X$. In such a case, the Lie algebra of all vector fields to X is canonically isomorphic to the Lie algebra of the group $\mathbf{Aut}\, X$. Hence, the vector field D

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corresponds to some element $\tilde{D}$ in the Lie algebra of $\mathbf{Aut}\, X$, that is to say, D is a fundamental vector field with respect to the action of $\mathbf{Aut}\, X$ on X. In fact, the former theorem holds when X is not complete, if D is assumed to be a fundamental field with respect to the action on X of some algebraic group.

The above mentioned commutative group G is defined to be the minimal tangent subvariety of $\tilde{D}$ through the identity of $\mathbf{Aut}\, X$.

We determine the group G in the case of vector fields on the complex projective space $\mathbb{P}_n$. It is well known that any vector field on $\mathbb{P}_n$ may be expressed as $D = \pi_*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$, where $\pi\colon \mathbb{C}^{n+1} \to \mathbb{P}_n$ is the natural projection. The group G is determined by two integers s, δ associated to D as follows. Let us consider the field $\mathbb{C}$ as an infinite dimensional affine space over $\mathbb{Q}$; then s is the dimension of the minimal affine $\mathbb{Q}$–subspace of $\mathbb{C}$ containing all the eigenvalues of the matrix (λ_{ij}). The integer δ is 0 when the matrix (λ_{ij}) is diagonalizable and 1 otherwise. We prove the following result:

Theorem. *The group G associated to a vector field $D = \pi_*(\sum \lambda_{ij} z_j \frac{\partial}{\partial z_i})$ on the complex projective space $\mathbb{P}_n$ is*

$$G \;=\; (\mathbb{G}_m)^s \times (\mathbb{G}_a)^\delta .$$

($\mathbb{G}_m$ and $\mathbb{G}_a$ are the multiplicative and additive lines respectively.)

Corollary. *The dimension of the minimal tangent subvarieties of D is $s+\delta$ (generically). The field of meromorphic functions on $\mathbb{P}_n$ which are first integrals of D has transcendence degree equal to $n - s - \delta$.*

1. Preliminaries

From now on, k will be an algebraically closed field of characteristic 0 and X will be a k–scheme.

1.1. **Differential algebras.** Let us recall some elementary facts about differential algebras. A **differential** k**–algebra** is a k–algebra A endowed with a k–linear derivation $D : A \to A$. An ideal I of A is said to be a **differential ideal** if $D(I) \subseteq I$.

Proposition 1.1. *The nilradical of any differential algebra A is a differential ideal.*

Proof. If $a^n = 0$, applying D^n we obtain $n!(Da)^n + ba = 0$, so that Da is nilpotent. □

Proposition 1.2. *Any minimal prime ideal of a differential noetherian k–algebra is a differential ideal.*

Proof. Replacing A by $A/\operatorname{rad} A$, we may assume that $\operatorname{rad} A = 0$. In such case, $0 = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal prime ideals of A. Let $0 \neq b \in \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$. If $a \in \mathfrak{p}_1$, then $ab = 0$ because $ab \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r = 0$. Hence $(Da)b = -aDb$, so that both terms are 0, since $aDb \in \mathfrak{p}_1$ and $bDa \in \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$. Finally, from $(Da)b = 0$ it follows that $Da \in \mathfrak{p}_1$ because $b \notin \mathfrak{p}_1$. Therefore $D\mathfrak{p}_1 \subseteq \mathfrak{p}_1$. □

1.2. **Functor of points.** The **functor of points** of X is defined to be the following contravariant functor on the category of k–schemes

$$X^\bullet(T) := \mathrm{Hom}_k(T, X) \ .$$

For any two k–schemes X, Y we have (Yoneda's lemma)

$$\mathrm{Hom}_k(X, Y) = \mathrm{Hom}_{\mathrm{funct.}}(X^\bullet, Y^\bullet) \ .$$

In particular, in order to define a morphism of k–schemes $X \to Y$ it is enough to construct a morphism of functors $X^\bullet \to Y^\bullet$. We shall use this fact later on.

Elements of $X^\bullet(T)$, that is to say, morphisms of k–schemes $x : T \to X$, are said to be T–**valued points** (or T–**points**) of X. Moreover, we use the following convention: Given a T-point $x : T \to X$ and a morphism of k–schemes $T' \to T$, the composition $T' \longrightarrow T \xrightarrow{x} X$ will be denoted also by x, that is to say, we do not change the notation of a point after a basis change.

1.3. **Infinitesimal automorphisms.** Let $k[\varepsilon] = k[x]/(x^2)$ and let $i \colon X \hookrightarrow X_{k[\varepsilon]} = X \times_k k[\varepsilon]$ be the closed immersion defined by $\varepsilon = 0$.

An **infinitesimal automorphism** of X is defined to be an automorphism of $k[\varepsilon]$–schemes $\tau_\varepsilon \colon X_{k[\varepsilon]} \to X_{k[\varepsilon]}$ satisfying the infinitesimal condition

$$(\tau_\varepsilon)_{|\varepsilon=0} = \mathrm{id}_X \ ,$$

that is to say, $\tau_\varepsilon \circ i = i$.

Any vector field D on X determines an infinitesimal automorphism τ_ε of X given by the following morphism of $k[\varepsilon]$–algebras:

$$\tau_\varepsilon : \mathcal{O}_X[\varepsilon] \longrightarrow \mathcal{O}_X[\varepsilon], \qquad a \mapsto \tau_\varepsilon(a) = a + \varepsilon Da$$

and, conversely, any infinitesimal automorphism is clearly defined by a unique vector field.

1.4. **Tangent subschemes.** Let D be a vector field on X. A closed subscheme Y of X, defined by a sheaf of ideals I of $\mathcal{O}_X$, is said to be a **tangent subscheme** of D if $D(I) \subseteq I$. This condition implies that D induces a derivation of the sheaf $\mathcal{O}_Y = \mathcal{O}_X/I$, so that it defines a vector field on Y.

A closed subscheme Y of X is a tangent subscheme of D if and only if its functor of points $Y^\bullet$ is stable under the corresponding infinitesimal automorphism τ_ε, that is to say, for any k–scheme T and any point $x \in X^\bullet(T)$, we have

$$x \in Y^\bullet(T) \ \Rightarrow \ \tau_\varepsilon(x) \in Y^\bullet_{k[\varepsilon]}(T_{k[\varepsilon]}) = Y^\bullet(T_{k[\varepsilon]})$$

where $\tau_\varepsilon(x)$ stands for the composition $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]} \xrightarrow{\tau_\varepsilon} X_{k[\varepsilon]}$.

Given a closed point $x \in X$, we denote by Y_x the minimal tangent subscheme of D passing through x. Remark that $Y_{x'} \subseteq Y_x$ whenever $x' \in Y_x$.

From Propositions 1.1 and 1.2, it follows that Y_x *is reduced and irreducible.*

1.5. **Zeros of a vector field.** Let D be a vector field on X and let us consider the natural morphism $\Omega^1_X \xrightarrow{D} \mathcal{O}_X$, $\mathrm{d}a \mapsto Da$. The image of this morphism is a certain sheaf of ideals I of $\mathcal{O}_X$. The **subscheme of zeros** of D is the closed subscheme Z_D of X defined by the sheaf of ideals I.

The subscheme of zeros Z_D may be defined in terms of the infinitesimal automorphism τ_ε corresponding to the vector field D. In fact, the functor of τ_ε–invariant points of X is representable by the subscheme of zeros Z_D as follows.

Lemma 1.3. *We have*

$$Z_D^\bullet(T) \;=\; \{x \in X^\bullet(T) : \tau_\varepsilon(x) = x\}$$

where the equality $\tau_\varepsilon(x) = x$ *states the coincidence of the composition morphism* $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]} \xrightarrow{\tau_\varepsilon} X_{k[\varepsilon]}$ *with the morphism* $T_{k[\varepsilon]} \xrightarrow{x} X_{k[\varepsilon]}$.

Proof. A morphism $x : T \to X$ factors through Z_D if and only if the composition $\Omega^1_X \xrightarrow{D} \mathcal{O}_X \xrightarrow{x} \mathcal{O}_T$ vanishes. Now, this condition is equivalent to the coincidence of the two following morphisms:

$$\mathcal{O}_X[\varepsilon] \xrightarrow{\;\tau_\varepsilon\;} \mathcal{O}_X[\varepsilon] \xrightarrow{\;x\;} \mathcal{O}_T[\varepsilon] \qquad a \;\mapsto\; a + \varepsilon Da \;\mapsto\; x(a) + \varepsilon x(Da),$$

$$\mathcal{O}_X[\varepsilon] \xrightarrow{\qquad x \qquad} \mathcal{O}_T[\varepsilon] \qquad a \;\mapsto\; x(a)\ .$$

□

1.6. **Differential morphisms.** Let (A, D) and (A', D') be two differential k–algebras. A morphism of k-algebras $\varphi : A \to A'$ is said to be differential if it commutes with the given derivations: $\varphi(Da) = D'(\varphi(a))$.

Now let X and X' be two k–schemes and let D and D' be vector fields on X and X' respectively. A morphism of k–schemes $\varphi : X' \to X$ is said to be **differential** when the morphism of k–algebras $\varphi : \mathcal{O}_X(U) \to \mathcal{O}_{X'}(V)$ is differential, for any pair of open sets $U \subseteq X$ and $V \subseteq \varphi^{-1}(U)$.

Let τ_ε and τ'_ε be the infinitesimal automorphisms corresponding to D and D' respectively. A morphism of k–schemes $\varphi : X' \to X$ is differential if and only if the morphism $\varphi : X'_{k[\varepsilon]} \longrightarrow X_{k[\varepsilon]}$ satisfies $\varphi \circ \tau'_\varepsilon = \tau_\varepsilon \circ \varphi$.

The following statement is trivial.

Proposition 1.4. *Let* $\varphi : (X', D') \longrightarrow (X, D)$ *be a differential morphism of* k*–schemes. If* Y *is a tangent subscheme of* (X, D)*, then* $\varphi^{-1}(Y) = Y \times_X X'$ *is a tangent subscheme of* (X', D').

2. Algebraic group associated to a vector field

In this section X will be a proper k–scheme.

Definition 2.1. Let us consider the functor of automorphisms of X, defined on the category of k–schemes,

$$F(T) := \mathrm{Aut}_T(X \times_k T)\ .$$

This functor is representable (see [9]) by an algebraic group $\mathbf{Aut}\, X$, named **scheme of automorphisms** of X ([4]), that is to say,

$$(\mathbf{Aut}\, X)^\bullet(T) \;=\; \mathrm{Aut}_T(X \times_k T)\ .$$

Now let D be a vector field on X and let τ_ε be the corresponding infinitesimal automorphism. This automorphism induces an infinitesimal automorphism $\tilde\tau_\varepsilon$ of $\mathbf{Aut}\, X$. We construct this automorphism $\tilde\tau_\varepsilon : (\mathbf{Aut}\, X)_{k[\varepsilon]} \longrightarrow (\mathbf{Aut}\, X)_{k[\varepsilon]}$ by means of the functor of points: For any $k[\varepsilon]$–scheme T we define

$$\tilde\tau_\epsilon : \mathrm{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T) \longrightarrow \mathrm{Aut}_T(X_{k[\varepsilon]} \times_{k[\varepsilon]} T), \qquad g \mapsto \tau_\varepsilon \circ g.$$

Note that $\tilde\tau_\varepsilon$ satisfies the infinitesimal condition $(\tilde\tau_\varepsilon)_{|\varepsilon=0} = \mathrm{id}_{\mathbf{Aut}\, X}$.

The infinitesimal automorphism $\tilde{\tau}_\varepsilon$ corresponds to a certain tangent field $\tilde{D}$ on $\mathbf{Aut}\, X$. Note that, by definition, $\tilde{\tau}_\varepsilon$ commutes with right translations in $\mathbf{Aut}\, X$, so that the field $\tilde{D}$ is invariant under right translations.

Therefore, *any vector field D on a proper k–scheme X has a canonically associated vector field $\tilde{D}$ on the scheme of automorphisms $\mathbf{Aut}\, X$, which is invariant under right translations.*

Moreover, it may be proved that the map $D \mapsto \tilde{D}$ defines an isomorphism of the Lie algebra of vector fields on X onto the Lie algebra of the group $\mathbf{Aut}\, X$ (we shall not use this fact).

Definition 2.2. Let D be a vector field on a proper k–scheme X and let $\tilde{D}$ be the corresponding vector field on the scheme of automorphisms $\mathbf{Aut}\, X$, invariant under right translations. The **associated group** of D is defined to be the minimal tangent subscheme G of $\tilde{D}$ passing through the identity of $\mathbf{Aut}\, X$. We shall prove that G is a commutative algebraic subgroup of $\mathbf{Aut}\, X$.

Proposition 2.3. *G is a connected algebraic closed subgroup of $\mathbf{Aut}\, X$.*

Proof. By definition G is a closed subscheme of $\mathbf{Aut}\, X$. Moreover, G is integral (hence connected) by Propositions 1.1 and 1.2. Finally, we have to show that the product $G \times G \dot{\longrightarrow} \mathbf{Aut}\, X$ factors through G and that the inverse morphism $\mathbf{Aut}\, X \to \mathbf{Aut}\, X$, $g \mapsto g^{-1}$ takes G into itself. Since G is reduced, it is enough to prove both statements in the case of closed points; that is to say, it is enough to show that $G^\bullet(k)$ is a subgroup of $(\mathbf{Aut}\, X)^\bullet(k)$.

Let $g \in G^\bullet(k)$. Since G is the minimal tangent subvariety of $\tilde{D}$ through the identity and $\tilde{D}$ is invariant under right translations, it follows that $G \cdot g$ is the minimal tangent subvariety of $\tilde{D}$ passing through g. Hence $G{\cdot}g \subseteq G$, because G also passes through g. This inclusion is not strict, since otherwise, multiplying at right by g^n, we would obtain an infinite decreasing sequence of closed sets $G{\cdot}g^{n+1} \subset G{\cdot}g^n$, so contradicting the noetherian character of $\mathbf{Aut}\, X$. Hence $G \cdot g = G$ and then $G^\bullet(k) \cdot g = G^\bullet(k)$ for any $g \in G^\bullet(k)$. It follows readily that $G^\bullet(k)$ is a group. □

Proposition 2.4. *G is commutative.*

Proof. Let us consider the infinitesimal automorphism

$$\sigma : (\mathbf{Aut}\, X)_{k[\varepsilon]} \longrightarrow (\mathbf{Aut}\, X)_{k[\varepsilon]}$$

defined, in terms of the functor of points, by the formula $\sigma(g) = \tau_\varepsilon \circ g \circ \tau_\varepsilon^{-1}$. This infinitesimal automorphism corresponds to a certain vector field on $\mathbf{Aut}\, X$. Let H be the corresponding *closed* subscheme of zeros. By Lemma 1.3, the functor of points of H is

$$H^\bullet(T) = \{g \in (\mathbf{Aut}\, X)^\bullet(T) : \sigma(g) = g\} = \{g \in (\mathbf{Aut}\, X)^\bullet(T) : \tau_\varepsilon \circ g = g \circ \tau_\varepsilon\} .$$

Remark that $H^\bullet(T)$ is a subgroup of $(\mathbf{Aut}\, X)^\bullet(T)$, hence H is a closed algebraic subgroup of $\mathbf{Aut}\, X$. Let us consider the center C of H, which is a commutative closed subgroup. The functor of points of the center is

$$C^\bullet(T) = \{c \in H^\bullet(T) :\ c \circ h = h \circ c \ \text{ for any } h \in H^\bullet(T') \text{ and any } T' \to T\}$$

(we refer to [2], II, §1 3.7 for the existence of the center). Let us prove that C is a tangent subscheme of $\tilde{D}$. According to 1.4, we have to show that the functor $C^\bullet$ is stable by the automorphism $\tilde{\tau}_\varepsilon$: If $c \in C^\bullet(T)$, let us prove that

$\tilde{\tau}_\varepsilon(c) = \tau_\varepsilon \circ c \in C^\bullet(T_{k[\varepsilon]})$; in fact, for any morphism $T' \to T_{k[\varepsilon]}$ and any point $h \in H^\bullet(T')$ we have

$$(\tau_\varepsilon \circ c) \circ h = \tau_\varepsilon \circ (c \circ h) = \tau_\varepsilon \circ (h \circ c) = h \circ (\tau_\varepsilon \circ c) \ .$$

Finally, G is contained in C, because C is a tangent subscheme of $\tilde{D}$ passing through the identity and G is minimal. Since C is commutative, so is G. □

Note 2.5. A more general result than Proposition 2.4 was proved by Chevalley [1] in the realm of Lie algebras. In fact, if L is a Lie subalgebra of the Lie algebra of an algebraic group and L_{alg} stands for the minimal algebraic Lie subalgebra (in the sense that it is the Lie algebra of an algebraic subgroup) containing L, then [1], Chap. II, Th. 13, states that $[L, L] = [L_{\mathrm{alg}}, L_{\mathrm{alg}}]$; hence L_{alg} is abelian whenever L is. In particular, if $L = \langle D \rangle$, then L_{alg} is an abelian Lie subalgebra.

Let us consider the structure of the associated algebraic group G. According to the fundamental structure theorem of algebraic groups (see [12] or [13]), G has an affine connected normal subgroup N such that the quotient $A = G/N$ is an abelian variety. Moreover, any connected commutative affine group is a direct product of multiplicative and additive lines: $N = \mathbb{G}_m^r \times \mathbb{G}_a^\delta$. In conclusion, G is an extension of an abelian variety by a direct product of multiplicative and additive lines:

$$0 \longrightarrow \mathbb{G}_m^r \times \mathbb{G}_a^\delta \longrightarrow G \longrightarrow A \longrightarrow 0 \ .$$

When this extension is trivial (G being the associated group), one may easily prove that $\delta \leq 1$.

3. Tangent subschemes

Let us recall the notation of the former section: X is a proper scheme over an algebraically closed field k of characteristic zero, D is a vector field on X and G is the associated group. Recall that G is a subgroup of the group $\mathbf{Aut}\, X$, so that we have an obvious action $\mu : G \times X \to X$.

Let $\tilde{D}$ be the right-invariant vector field on $\mathbf{Aut}\, X$ induced by D, and let $\tilde{\tau}_\varepsilon$ be the corresponding infinitesimal automorphism: $\tilde{\tau}_\varepsilon(g) = \tau_\varepsilon \circ g$. Let us consider $\tilde{D}$ as a vector field on G (it may be done because G is a tangent subscheme of $\tilde{D}$). Then we may consider on $G \times X$ the vector field $(\tilde{D}, 0)$, the corresponding infinitesimal automorphism being $(\tilde{\tau}_\varepsilon, Id)$.

Lemma 3.1. *The natural action $\mu : G \times X \to X$ is a differential morphism. As well, for any closed point $x \in X$ the morphism $\mu_x : G = G \times \{x\} \subset G \times X \xrightarrow{\mu} X$ is also differential.*

Proof. We have to show that the action μ commutes with the respective infinitesimal automorphisms: $\mu \circ (\tilde{\tau}_\varepsilon, Id) = \tau_\varepsilon \circ \mu$. We prove it by means of the functor of points: Let $(g, x) \in G^\bullet(T) \times X^\bullet(T) = (G \times X)^\bullet(T)$, where T is a $k[\varepsilon]$–scheme; we have

$$(\mu \circ (\tilde{\tau}_\varepsilon, Id))(g, x) = \mu(\tau_\varepsilon \cdot g, x) = (\tau_\varepsilon \cdot g)(x) = \tau_\varepsilon(g(x)) = \tau_\varepsilon(\mu(g, x)) = (\tau_\varepsilon \circ \mu)(g, x) \ .$$

The second statement may be proved in a similar way. □

Theorem 3.2. *Let D be a vector field on a proper k–scheme X and let G be the associated group. For any closed point $x \in X$, the minimal tangent subscheme Y_x passing through x coincides with the closure of the orbit of x, that is,*

$$Y_x \ = \ \overline{G \cdot x} \ .$$

Proof. Let us consider the differential morphism $\mu_x : G \to X$, $g \mapsto g \cdot x$. By Proposition 1.4, $\mu_x^{-1}(Y_x)$ is a tangent subscheme of $\tilde{D}$, which contains the identity of G. Since G is minimal, we conclude that $G \subseteq \mu_x^{-1}(Y_x)$, hence $\mu_x(G) = G \cdot x \subseteq Y_x$ and it follows that $\overline{G \cdot x} \subseteq Y_x$.

In order to show the reverse inclusion $Y_x \subseteq \overline{G \cdot x}$, it is enough to show that $\overline{G \cdot x}$ is a tangent subscheme of D, because Y_x is minimal. Let g_0 be the generic point of G and let $x_0 = \mu_x(g_0)$. It is clear that x_0 is the generic point of $\overline{G \cdot x} = \overline{\mu_x(G)}$. Let $\mathfrak{p}_{x_0}$ be the maximal ideal of $\mathcal{O}_{X,x_0}$. The exact sequence

$$0 \longrightarrow \mathfrak{p}_{x_0} \longrightarrow \mathcal{O}_{X,x_0} \xrightarrow{\mu_x} \mathcal{O}_{G,g_0} \ ,$$

where the last morphism is differential, shows that $\mathfrak{p}_{x_0}$ is a differential ideal of $\mathcal{O}_{X,x_0}$. Let $\mathfrak{p}$ be the sheaf of ideals of the closed subscheme $\overline{G \cdot x}$. Since x_0 is the generic point of this subscheme, for any open set U in X we have

$$\mathfrak{p}(U) \ = \ \mathcal{O}_X(U) \cap \mathfrak{p}_{x_0}$$

(rigorously, $\mathfrak{p} = h^{-1}(\mathfrak{p}_{x_0})$ where $h : \mathcal{O}_X(U) \to \mathcal{O}_{X,x_0}$ is the natural morphism). It readily follows that $\mathfrak{p}(U)$ is a differential ideal of $\mathcal{O}_X(U)$. □

Proposition 3.3. *Let $G \times X \to X$ be an action of a connected algebraic group G over an integral quasi–projective k–scheme X. There is a G–invariant dense open set U in X such that the geometric quotient $\pi : U \to U/G$ exists.*

We shall prove this result in the Appendix. Putting Theorem 3.2 and Proposition 3.3 together we obtain the following result.

Theorem 3.4. *Let D be a vector field on an integral projective k–scheme X and let G be the associated group. There exists a dense open set U in X such that:*

a) *U is G–invariant and orbits of closed points in U are just minimal tangent subvarieties of the vector field D on U.*

b) *The geometric quotient $\pi : U \to Z = U/G$ exists. The sheaf $\mathcal{O}_Z$ coincides with the sheaf of first integrals of D, that is, for any open set V in Z we have*

$$\Gamma(V, \mathcal{O}_Z) \ = \ \{f \in \Gamma(\pi^{-1}V, \mathcal{O}_X) : Df = 0\} \ .$$

Proof. a) Let U be the open set whose existence states Proposition 3.3. The orbit of any closed point in U is a closed subset of U, since it is the fibre of $\pi : U \to U/G$ over a closed point of U/G. Applying Theorem 3.2 to the field D on U, we conclude that such orbits are just the minimal tangent subvarieties.

b) Since $\pi : U \to Z = U/G$ is a geometric quotient, the sheaf $\mathcal{O}_Z$ coincides with the sheaf of G–invariant functions,

$$\Gamma(V, \mathcal{O}_Z) \ = \ \Gamma(\pi^{-1}V, \mathcal{O}_X)^G \ ,$$

so that we have to show that a function $f \in \Gamma(\pi^{-1}V, \mathcal{O}_X)$ is G–invariant if and only if $Df = 0$.

Given a closed point $x \in \pi^{-1}V$, let us consider the minimal tangent subvariety $Y_x = G \cdot x$ of D passing through x. If f is G–invariant, then $f|_{Y_x}$ is constant, so that $(Df)(x) = (D(f|_{Y_x}))(x) = 0$.

Conversely, if $Df=0$, then we consider the ideal $I=(f-\lambda)$, where $\lambda := f(x)$. It is a differential ideal, hence it defines a tangent closed subscheme T of D. Since $x \in T$ and Y_x is minimal, we have $Y_x = G\cdot x \subseteq T$, hence $f=\lambda$ on $Y_x = G\cdot x$. Therefore f is constant on the orbits and we conclude that f is G–invariant. □

Remark 3.5. The above theorem holds when X is not complete if D is assumed to be a fundamental vector field with respect to the action $\mu\colon \mathcal{G}\times X \to X$ of some algebraic group $\mathcal{G}$, i.e, $D=\mu_*(\tilde{D},0)$ for some right-invariant vector field $\tilde{D}$ on $\mathcal{G}$. Recalling Definition 2.2, the **associated group** G to the vector field D is defined to be the minimal tangent subvariety of $\tilde{D}$ passing through the identity of $\mathcal{G}$. The results of Sections 2 and 3 remain valid, so that the associated group G is a commutative connected algebraic subgroup of $\mathcal{G}$, and the minimal tangent subvarieties of D are the closure of the orbits of G on X.

Remark 3.6. Theorem 3.4 may be generalized to a distribution generated by global vector fields. Let X be an integral projective k-scheme and let L be a vector subspace of the space of all global vector fields on X.

A closed subscheme of X is said to be a **tangent subscheme of** L if it is a tangent subscheme of any $D \in L$. Recall that each vector field D on X corresponds with a right-invariant vector field $\tilde{D}$ on $\mathbf{Aut}\,X$. Let $\tilde{L}$ be the space of all right-invariant vector fields $\tilde{D}$ on $\mathbf{Aut}\,X$ such that $D \in L$.

The results of Sections 2 and 3 (and their proofs) may be extended for the distribution L:

1. *The minimal tangent subscheme of $\tilde{L}$, passing through the identity of* $\mathbf{Aut}\,X$, *is a connected closed algebraic subgroup G of* $\mathbf{Aut}\,X$.
2. *If L is an abelian Lie algebra, then G is a commutative group.*
3. *The minimal tangent subscheme Y_x of L passing through a closed point $x \in X$ is the closure of the orbit, $Y_x = \overline{G\cdot x}$.*
4. *There exists a dense G-invariant open subset U in X such that:*
   a) *Orbits of closed points in U are just minimal tangent subvarieties of L on U;*
   b) *The geometric quotient $\pi : U \to Z = U/G$ exists and the sheaf $\mathcal{O}_Z$ coincides with the sheaf of first integrals of L, that is, for any open set V in Z we have*

$$\Gamma(V,\mathcal{O}_Z) \;=\; \{f \in \Gamma(\pi^{-1}V,\mathcal{O}_X) : Df=0 \;\text{ for any }\; D \in L\}.$$

4. Vector fields on $\mathbb{P}_n$

The aim of this section is to calculate the associated algebraic group of any vector field on the complex projective space $\mathbb{P}_n$.

Lemma 4.1. *Let us consider the differential algebras*

$$\mathbb{C}[z_1,\dots,z_r],\; D=\sum \mu_i z_i \frac{\partial}{\partial z_i},$$
$$\mathbb{C}[z_0,z_1,\dots,z_r],\; D=\frac{\partial}{\partial z_0}+\sum_{i>0}\mu_i z_i\frac{\partial}{\partial z_i}$$

where $\mu_1,\dots,\mu_r \in \mathbb{C}$. If $\mu_1,\dots,\mu_r$ are linearly independent over $\mathbb{Q}$, then any non-null differential ideal of these algebras contains some monomial $z_1^{a_1}\cdots z_r^{a_r}$.

Proof. The same argument holds in both algebras. It is easy to check that monomials $\lambda z_1^{a_1}\cdots z_r^{a_r}$ are the only eigenvectors of D. Let E_n be the vector subspace of all polynomials of degree $\leq n$ and note that $D(E_n)\subseteq E_n$ in both cases. Let I be a non-null differential ideal. It is clear that $I\cap E_n\neq 0$ when $n\gg 0$. Since the dimension of $I\cap E_n$ is finite, the linear map $D: I\cap E_n\longrightarrow I\cap E_n$ has some eigenvector, hence I contains some monomial. □

Corollary 4.2. *If $\mu_1,\dots,\mu_r$ are linearly independent over $\mathbb{Q}$, then the differential algebras*

$$\mathbb{C}[z_1,\dots,z_r,(z_1\cdots z_r)^{-1}],\ D=\sum\mu_i z_i\frac{\partial}{\partial z_i},$$
$$\mathbb{C}[z_0,z_1,\dots,z_r,(z_1\cdots z_r)^{-1}],\ D=\frac{\partial}{\partial z_0}+\sum_{i>0}\mu_i z_i\frac{\partial}{\partial z_i}$$

have no non-trivial differential ideal.

Proposition 4.3. *If $\mu_1,\dots,\mu_r\in\mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the following differential algebras of holomorphic functions on $\mathbb{C}$,*

$$\mathbb{C}[e^{\mu_1 t},\dots,e^{\mu_r t},e^{-\sum\mu_i t}],\quad \mathbb{C}[t,e^{\mu_1 t},\dots,e^{\mu_r t},e^{-\sum\mu_i t}]$$

(endowed with the derivation $\frac{\partial}{\partial t}$) have no non-trivial differential ideal.

Proof. Let us consider the obvious differential epimorphisms

$$\mathbb{C}[z_1,\dots,z_r,\frac{1}{z_1\cdots z_r}]\longrightarrow\mathbb{C}[e^{\mu_1 t},\dots,e^{\mu_r t},e^{-\sum\mu_i t}],$$
$$\mathbb{C}[z_0,z_1,\dots,z_r,\frac{1}{z_1\cdots z_r}]\longrightarrow\mathbb{C}[t,e^{\mu_1 t},\dots,e^{\mu_r t},e^{-\sum\mu_i t}],$$

where the algebras on the left are endowed with the derivations considered in Corollary 4.2. These epimorphisms are isomorphisms because both have null kernel by Corollary 4.2. □

Remark 4.4. By the last isomorphism in the former proof, the holomorphic functions $t,e^{\mu_1 t},\dots,e^{\mu_r t}$ are algebraically independent whenever $\mu_1,\dots,\mu_r$ are linearly independent over $\mathbb{Q}$.

4.5. Let $M=(\lambda_{ij})$ be an $n\times n$ matrix with complex coefficients. Let us consider the **linear vector field**

$$D\ =\ \sum\lambda_{ij}z_j\frac{\partial}{\partial z_i}$$

on $\mathbb{C}^n$. The corresponding infinitesimal automorphism τ_ε of $\mathbb{C}^n$ is a $\mathbb{C}[\varepsilon]$–linear transformation with matrix $Id+\varepsilon M\equiv e^{\varepsilon M}$. This automorphism induces an infinitesimal automorphism $\tilde\tau_\varepsilon$ of the full linear group Gl_n,

$$(Gl_n)_{\mathbb{C}[\varepsilon]}\xrightarrow{\tilde\tau_\varepsilon}(Gl_n)_{\mathbb{C}[\varepsilon]}\qquad g\mapsto\tau_\varepsilon\circ g\ ,$$

whose corresponding vector field $\tilde D$ on Gl_n is obviously right-invariant. Note that D is a fundamental vector field with respect to the natural action $\mu\colon Gl_n\times\mathbb{C}^n\to\mathbb{C}^n$, since we have $\mu_*(\tilde D,0)=D$ because $\mu\circ(\tilde\tau_\varepsilon,Id)=\tau_\varepsilon\circ\mu$.

According to Remark 3.5, the **associated group** G of the linear vector field D is defined to be the minimal tangent subvariety of $\tilde D$ passing through the identity of Gl_n. We know that G is a commutative connected algebraic subgroup of Gl_n, and that the minimal tangent subvarieties of D are the closure of the orbits of G on $\mathbb{C}^n$.

Indeed, it may be proved that the associated group G of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ coincides with the so-called differential Galois group of the linear differential system $z_i(t)' = \sum_j \lambda_{ij} z_j(t)$. See Note 4.7 below.

Let $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ be the algebra of holomorphic functions on $\mathbb{C}$ generated by the coefficients of the matrix e^{Mt} and the function $\det(e^{-Mt})$. It is a differential algebra with the derivation $\frac{\partial}{\partial t}$.

Lemma 4.6. *The group G associated to a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ is*

$$G = \operatorname{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$$

where $M = (\lambda_{ij})$.

Proof. Let $Gl_n = \operatorname{Spec} \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}]$. The infinitesimal automorphism $\tilde{\tau}_\varepsilon$ is

$$\tilde{\tau}_\varepsilon((x_{ij})) = \tau_\varepsilon \circ (x_{ij}) = e^{M\varepsilon} \circ (x_{ij}) = (Id + \varepsilon M) \circ (x_{ij}) = (x_{ij}) + \varepsilon M \circ (x_{ij}),$$

so that the derivation $\tilde{D}$ is (in matrix form)

$$(*) \qquad (\tilde{D} x_{ij}) = M \circ (x_{ij}).$$

Let us consider the obvious epimorphism

$$\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \longrightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$$

which transforms the coefficients of the universal matrix (x_{ij}) into the coefficients of the matrix e^{Mt}. Equality $(*)$ states that it is a differential morphism when the first algebra is endowed with the derivation $\tilde{D}$ and the second algebra is endowed with $\frac{\partial}{\partial t}$. Therefore $\operatorname{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is a tangent subscheme of $\tilde{D}$. Clearly, this closed subscheme passes through the identity of Gl_n when $t = 0$. In order to show that it is the minimal tangent subscheme, it is enough to show that the differential algebra $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ has no non-trivial differential ideal.

After a linear coordinate change, we may assume that the matrix M is in Jordan form. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M, then it is easy to check that

$$\mathbb{C}[e^{Mt}, \det(e^{-Mt})] = \mathbb{C}[\delta t, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}, e^{-\sum \lambda_i t}]$$

where $\delta = 0$ if M is diagonalizable and $\delta = 1$ otherwise.

Let $\mu_1, \ldots, \mu_r$ be a base of the $\mathbb{Q}$–vector space $\mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n \subset \mathbb{C}$, that is to say, $\mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n = \mathbb{Q}\mu_1 \oplus \cdots \oplus \mathbb{Q}\mu_r$. It is easy to check that

$$\mathbb{C}[\delta t, e^{\lambda_1 t}, \ldots, e^{\lambda_n t}, e^{-\sum \lambda_i t}] = \mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_j t}] .$$

Now, by Proposition 4.3, we conclude that $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ has no non-trivial differential ideal. □

Note 4.7. Let us explain the relation of the associated group G with the Galois theory of differential equations.

We have shown in the former proof that $\mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is a simple differential ring, i.e., it has no non-trivial differential ideal. According to ([10], Def. 1.15), this ring is the *Picard-Vessiot ring* of the differential equation $z' = Mz$, since e^M is a *fundamental matrix*. Let L be the field of fractions of the Picard–Vessiot ring, which is named the *Picard–Vessiot field* of the equation $z' = Mz$ ([10], Def. 1.21). Now, by Lemma 4.6, L is the field of functions of the closed algebraic subgroup $(G, \tilde{D}) \subset (Gl_n, \tilde{D})$. The action of G on itself by right–translations induces an action of G on L by differential automorphisms. It is immediate to check that any G-invariant function is constant: $L^G = \mathbb{C}$. Then, by the Galois correspondence ([10], Prop. 1.34), we conclude that G is the group of all differential automorphisms of L, i.e., G is the *differential Galois group* of the equation $z' = Mz$.

The following theorem improves a statement of ([8], Prop. 3.27), about the differential Galois group of a linear differential system with constant coefficients.

Theorem 4.8. *The associated group of a linear vector field* $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ *on* $\mathbb{C}^n$ *is*

$$G \;=\; \mathbb{G}_m^r \times \mathbb{G}_a^\delta$$

where r *stands for the dimension of the* $\mathbb{Q}$*–vector space* $\mathbb{Q}\lambda_1 + \cdots + \mathbb{Q}\lambda_n \subset \mathbb{C}$ *spanned by the eigenvalues of the matrix* $M = (\lambda_{ij})$*, and* $\delta = 0$ *when* M *is diagonalizable and* $\delta = 1$ *otherwise.*

Proof. Again we put $Gl_n = \operatorname{Spec} \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}]$. The group law in Gl_n is determined by the coproduct

$$\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \;\hookrightarrow\; \mathbb{C}[y_{ij}, \det(y_{ij})^{-1}] \otimes \mathbb{C}[z_{ij}, \det(z_{ij})^{-1}], \qquad (x_{ij}) = (y_{ij}) \circ (z_{ij}) \,.$$

The group law in $G = \operatorname{Spec} \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$ is defined by the induced coproduct in the quotient algebra $\mathbb{C}[x_{ij}, \det(x_{ij})^{-1}] \longrightarrow \mathbb{C}[e^{Mt}, \det(e^{-Mt})]$, that is,

$$\mathbb{C}[e^{Mt}, \det(e^{-Mt})] \;\hookrightarrow\; \mathbb{C}[e^{Mu}, \det(e^{-Mu})] \otimes \mathbb{C}[e^{Mv}, \det(e^{-Mv})],$$
$$e^{Mt} = e^{Mu} \circ e^{Mv} = e^{M(u+v)}.$$

In other words, this coproduct takes each function $f(t)$ into $f(u+v)$.

As shown in the proof of Lemma 4.6, the coordinate ring of G has the form

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_j t}] \,.$$

Recall that the functions $t, e^{\mu_1 t}, \ldots, e^{\mu_r t}$ are algebraically independent (Remark 4.4). With the coproduct $f(t) \mapsto f(u+v)$, the Hopf algebra

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_j t}]$$

has an obvious decomposition as a tensor product of Hopf algebras:

$$\mathbb{C}[\delta t, e^{\mu_1 t}, \ldots, e^{\mu_r t}, e^{-\sum \mu_j t}] \;=\; \mathbb{C}[\delta t] \otimes \mathbb{C}[e^{\mu_1 t}, e^{-\mu_1 t}] \otimes \cdots \otimes \mathbb{C}[e^{\mu_r t}, e^{-\mu_r t}] \,,$$

hence $G = \mathbb{G}_a^\delta \times \mathbb{G}_m \times \overset{r}{\cdots} \times \mathbb{G}_m$. □

Remark 4.9. By Theorem 3.4a, there exists a G–invariant dense open set U in $\mathbb{C}^n$ such that the orbits of G in U coincide with the minimal tangent subvarieties of D in U. The isotropy subgroup of any point of U is the identity subgroup. Let us give a (summarized) proof of this fact: Every (flat) family of subgroups of $G = \mathbb{G}_m^r \times \mathbb{G}_a^\delta$ is a constant family, hence all the points of U have the same isotropy subgroup (generically); since G acts faithfully on U, we conclude that such a subgroup is the identity. Therefore, the orbits of G in U have the same dimension as G, that is to say:

Minimal tangent subvarieties of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ *on* $\mathbb{C}^n$ *have dimension* $r + \delta$ *(generically).*

By 3.4b, the field of rational functions on the quotient variety $Z = U/G$ coincides with the field of rational functions on $\mathbb{C}^n$ which are first integrals of D. Since $\dim Z = \dim U - \dim G = n - r - \delta$, we conclude that:

The field of all rational functions on $\mathbb{C}^n$ which are first integrals of a linear vector field $D = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ has transcendence degree $n - r - \delta$.

This field may be computed by methods of Linear Algebra [3].

4.10. Let $\mathbb{P}_n$ be the n–dimensional projective space and let $\pi : \mathbb{C}^{n+1} \longrightarrow \mathbb{P}_n$ be the natural projection. It is well known that any vector field D on $\mathbb{P}_n$ is the projection by π of some linear vector field $D_0 = \sum \lambda_{ij} z_j \frac{\partial}{\partial z_i}$ on $\mathbb{C}^{n+1}$. Moreover, such a linear field is unique up to the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$. These facts follow readily from the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_n} \longrightarrow Hom_{\mathcal{O}_{\mathbb{P}_n}}(\mathcal{O}_{\mathbb{P}_n}(-1), \mathcal{O}_{\mathbb{P}_n}^{n+1}) \longrightarrow \mathcal{D}_{\mathbb{P}_n} \longrightarrow 0 ,$$

where $\mathcal{O}_{\mathbb{P}_n}(-1)$ is the sheaf of sections of the tautological line bundle on $\mathbb{P}_n$, $\mathcal{D}_{\mathbb{P}_n}$ is the sheaf of vector fields on $\mathbb{P}_n$ and $Hom_{\mathcal{O}_{\mathbb{P}_n}}(\mathcal{O}_{\mathbb{P}_n}(-1), \mathcal{O}_{\mathbb{P}_n}^{n+1})$ is the sheaf of π-projectable vector fields.

Theorem 4.11. *Let $D = \pi_*(\sum \lambda_{ij} z_i \frac{\partial}{\partial z_i})$ be a vector field on $\mathbb{P}_n$. Its associated group is*

$$G = \mathbb{G}_m^s \times \mathbb{G}_a^\delta$$

where s stands for the dimension of the minimal $\mathbb{Q}$–affine subspace of $\mathbb{C}$ containing all the eigenvalues of the matrix (λ_{ij}), and $\delta = 0$ when such a matrix is diagonalizable and $\delta = 1$ otherwise.

Proof. Let $D_0 = \sum \lambda_{ij} z_i \frac{\partial}{\partial z_i}$. After the addition of a vector field proportional to $\sum z_i \frac{\partial}{\partial z_i}$, we may assume that the matrix $M = (\lambda_{ij})$ has the eigenvalue 0. In such case the minimal $\mathbb{Q}$–affine subspace containing the eigenvalues of (λ_{ij}) is a $\mathbb{Q}$–vector space. By Theorem 4.8, the group associated to D_0 is

$$G_0 = \mathbb{G}_m^s \times \mathbb{G}_a^\delta .$$

Let $0 \neq v \in \mathbb{C}^{n+1}$ be an eigenvector of M of eigenvalue 0. Note that v is a fixed point of the infinitesimal automorphism τ_ε^0 (corresponding to D_0) since $\tau_\varepsilon^0(v) = e^{\varepsilon M}(v) = (Id + \varepsilon M)(v) = v$.

Let H_v be the stabilizer of v, which is a closed algebraic subgroup of Gl_{n+1}. The functor of points of H_v is

$$H_v^\bullet(T) = \{g \in Gl_{n+1}^\bullet(T) : g(v) = v\} .$$

We have that H_v is a tangent subvariety of $(Gl_{n+1}, \tilde{D}_0)$, since

$$g \in H_v^\bullet(T) \Rightarrow \tilde{\tau}_\varepsilon^0(g) = \tau_\varepsilon^0 \circ g \in H_v^\bullet(T) .$$

Since G_0 is minimal we obtain that $G_0 \subseteq H_v$.

Analogously, denoting by H_p the stabilizer of $p = \pi(v) \in \mathbb{P}_n$ with respect to the action of $PGl_{n+1} = \mathbf{Aut}\, \mathbb{P}_n$, we may prove that H_p is a tangent subvariety of $(PGl_{n+1}, \tilde{D})$ and then $G \subseteq H_p$.

Now, it is immediate that the natural epimorphism $Gl_{n+1} \to PGl_{n+1}$ induces a differential isomorphism $(H_v, \tilde{D}_0) \longrightarrow (H_p, \tilde{D})$. Via this isomorphism, we conclude that $G_0 = G$. □

Remark 4.12. The same arguments used in Remark 4.9 show that:

Minimal tangent subvarieties of a vector field D on $\mathbb{P}_n$ have dimension $s + \delta$ (generically).

The field of all rational functions on $\mathbb{P}_n$ which are first integrals of a vector field has transcendence degree $n - s - \delta$. (The case $n = 2$ is well known; see [7], pp. 12-16.)

5. A counterexample

Without the hypothesis of X being complete, it does not follow the existence, for any vector field D on X, of a dense open set U and a projection $\varphi : U \to Z$ whose fibres are the minimal tangent subvarieties of D in U.

As a counterexample, let us consider the field

$$D \ = \ z_1 z_4 \frac{\partial}{\partial z_4} + z_2 z_5 \frac{\partial}{\partial z_5} + z_3 z_6 \frac{\partial}{\partial z_6}$$

on $X = \mathbb{C}^6$. It is clear that D is tangent to any 3–plane $z_1 = \lambda_1, z_2 = \lambda_2, z_3 = \lambda_3$. On these planes D is a linear vector field with associated group $\mathbb{G}_m^r$, where r is the dimension of the $\mathbb{Q}$–vector space $\mathbb{Q}\lambda_1 + \mathbb{Q}\lambda_2 + \mathbb{Q}\lambda_3$ (see Theorem 4.8). In each 3–plane, the minimal tangent subvarieties generically have dimension r (Remark 4.9). Now, points $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ with $r = 3$ form a dense set in $\mathbb{C}^3$ (and so do points with $r = 2$). Therefore, minimal tangent subvarieties of dimension 3, as well as those of dimension 2, form a dense subset of $\mathbb{C}^6$. This fact prevents the existence of such projection $\varphi : U \to Z$, due to the semicontinuity of the dimension of the fibres of any algebraic morphism.

6. The case of positive characteristic

In this section, k denotes a field of characteristic $p > 0$. Let $\mathbb{G}_a = \operatorname{Spec} k[t]$ be the additive line. The subscheme $G := \operatorname{Spec} k[t]/(t^p)$ is a finite infinitesimal subgroup of $\mathbb{G}_a$.

Proposition 6.1. *Let X be a k–scheme. Each vector field D on X defines an action $\mu\colon G \times_k X \to X$,*

$$\mu^* \colon \mathcal{O}_X \to \mathcal{O}_X[t]/(t^p), \qquad \mu^*(f) = \sum_{r=0}^{p-1} \frac{1}{r!} D^r(f) t^r \ .$$

Conversely, any action $\mu\colon G \times_k X \to X$ is defined by a unique vector field D on X.

The proof of Proposition 6.1 is a simple exercise.

Let D be a vector field on X and let $\mu\colon G \times_k X \to X$ be the corresponding action. If Y is a tangent subscheme of (X, D), then the corresponding action $G \times_k Y \to Y$ is the restriction to Y of the action of G on X. Therefore, a closed subscheme Y of X is tangent to D if and only if it is G–invariant. We conclude that the minimal tangent subscheme passing through a closed point x is the orbit $G \cdot x =$ scheme–theoretic image of $G \times x \subseteq G \times X \xrightarrow{\mu} X$, that is to say, $G \cdot x$ is the closed subscheme of X defined by the ideal of all functions f such that $f(x) = Df(x) = \cdots = D^{p-1} f(x) = 0$. Note that each orbit has a unique point, but it is not a reduced scheme in general.

Appendix A. Quotients by algebraic groups

Theorem A.1 (Rosenlicht [11]). *Let $\mu\colon G\times_k X\to X$ be an action of an affine algebraic group G on an integral quasi–projective variety X. There exists a G–invariant dense open subset $U\subseteq X$ such that the geometric quotient $U\to U/G$ exists.*

The purpose of this Appendix is to extend Rosenlicht's result to the case of an algebraic group G non-necessarily affine.

Quotients by abelian varieties. Let $\mu\colon A\times_k X\to X$ be an action of an abelian variety A on an integral quasi–projective variety X.

Lemma A.2. *The action $\mu\colon A\times_k X\to X$, $(a,x)\mapsto a\cdot x$, and the morphism $\phi\colon A\times_k X\to X\times_k X$, $(a,x)\mapsto(a\cdot x,x)$, are projective morphisms.*

Proof. The isomorphism $\varphi\colon A\times X\to A\times X$, $(a,x)\mapsto(a,a\cdot x)$, makes commutative the triangle

$$\begin{array}{ccc} A\times X & \xrightarrow{\varphi} & A\times X \\ {\scriptstyle\mu}\searrow & & \swarrow{\scriptstyle p_2} \\ & X & \end{array}$$

Since any abelian variety is projective, the map $p_2\colon A\times X\to X$ is a projective morphism and then the above commutative triangle implies that $\mu\colon A\times X\to X$ is also a projective morphism. Finally, $\phi=\mu\times p_2$ is a projective morphism because μ and p_2 are also. □

Let R be the image of the map $\phi\colon A\times_k X\to X\times_k X$, $(a,x)\mapsto(a\cdot x,x)$, that is to say, R is the equivalence relation defined by the action of A over X. By the previous lemma, R is a closed subset of $X\times_k X$. We shall consider R as a closed subscheme of $X\times_k X$ with the reduced structure.

Lemma A.3. *The projection $p_1\colon R\to X$ is a projective morphism.*

Proof. Since $\phi\colon A\times X\to R$ is surjective and the composition morphism $\mu=p_1\circ\phi: A\times X\to R\to X$ is proper, it is easy to check that the valuative criterion of properness ([6], II, Th. 4.7) holds for the morphism $p_1\colon R\to X$.

Moreover, $R\subseteq X\times X\xrightarrow{p_1}X$ is a quasi-projective morphism, hence we conclude that $p_1\colon R\to X$ is a projective morphism. □

Given an A–stable open subset U of X, we write $R_U:=p_1^{-1}(U)$, i.e., R_U is the equivalence relation defined by the action of A over U.

Lemma A.4. *There exists an A–invariant dense open subset U of X such that $p_1\colon R_U\to U$ is a flat morphism.*

Proof. By the semicontinuity character of the Hilbert polynomial of the fibres of a projective morphism $p_1\colon R\to X$, we have that the subset U of all points $x\in X$, whose fibre $p_1^{-1}(x)$ has the same Hilbert polynomial than the fibre of the generic point, is a dense open subset of X. This open subset U is invariant by the action of A over X, since the projection $p_1\colon R\to X$ is an A–equivariant morphism (the action of A over R is defined by the formula $a\cdot(x_1,x_2)=(a\cdot x_1,x_2)$).

Finally, since the fibres of $p_1\colon R_U\to U$ have the same Hilbert polynomial, we conclude (see [6], III, Th. 9.9) that $p_1\colon R_U\to U$ is a flat morphism. □

Theorem A.5. *Let $\mu\colon A \times_k X \to X$ be an action of an abelian variety A on an integral quasi-projective variety X. There exists an A–invariant dense open subset U of X such that the geometric quotient $\pi\colon U \to U/A$ exists.*

Proof. By the previous lemmas, there exists an A–invariant dense open subset U of X such that $p_1\colon R_U \to U$ is a projective flat morphism. By a theorem of Grothendieck ([5], V, Th. 7.1), there exists the quotient map $\pi\colon U \to U/A$. □

The general case.

Theorem A.6. *Let $\mu\colon G \times_k X \to X$ be an action of a connected smooth group G on an integral quasi–projective variety X. There exists a G–invariant dense open subset $U \subseteq X$ such that the geometric quotient $U \to U/G$ exists.*

Proof. By the structure theorem of algebraic groups [12], there exists a normal affine subgroup G_0 of G such that $A = G/G_0$ is an abelian variety. By Rosenlicht's result, there exists a G_0–invariant dense open subset $U_0 \subseteq X$ such that the geometric quotient $\pi\colon U_0 \to U_0/G_0$ exists. Taking $G \cdot U_0$ instead of U_0, we may assume that U_0 is G–invariant. By Theorem A.5, there exists an A–invariant dense open subset V in U_0/G_0 such that the geometric quotient $V \to V/A$ exists. Then $U := \pi^{-1}V$ is the desired open set, since $U/G = (U/G_0)/A = V/A$. □

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Departamento de Matematicas, Universidad de Extremadura, Av. de Elvas s/n, Badajoz 06071, Spain
E-mail address: jsancho@unex.es